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Supercritical withdrawal from a two-layer fluid through a line sink

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Accurate numerical solutions to the problem of finding the location of the interface between two unconfined regions of fluid of different density during the withdrawal process are presented. Supercritical flows are considered, in which the interface is drawn directly into the sink. As the flow rate is reduced, the interface enters the sink more steeply, until the solution method breaks down just before the interface enters the sink vertically from above, and becomes flow from the lower layer only. This lower bound on supercritical flow is compared with the upper bound on single-layer (free surface) flow with good agreement.

1. Introduction

The withdrawal of water from a fluid consisting of several layers of different density has a range of engineering applications. In particular, understanding the process of withdrawal from reservoirs is important for achieving various water quality constraints, and in modelling the physical processes within the water body. Withdrawal and inflow to solar ponds are used to extract energy and to control the stratification in the pond in order to maintain stability and optimize efficiency. In power station cooling ponds, efficient operation can be maintained by paying careful attention to withdrawal (Imberger & Hamblin 1982).

Experiments (Gariel 1949; Harleman & Elder 1965; Jirka 1979; Hocking 1991*b*) show that the qualitative behaviour of the flow when withdrawal occurs through a narrow horizontal slot in the vertical endwall of a rectangular tank containing two homogeneous layers of different density is as follows. If the slot is situated within the lower layer, buoyancy forces ensure that only fluid from the lower layer is drawn through the slot at low values of the flow rate. At this time, there is a very slow (compared to the flow in the lower layer) circulation of the fluid in the upper layer. The interface between the two layers remains approximately horizontal once the transient wave motion caused by opening the slot has dissipated, except for a slight thickening of the interface near the wall directly above the slot. As the depth of the lower layer decreases the effective flow rate increases, and at some critical value the interface is suddenly pulled down and enters the slot directly, so that the upper layer also begins to flow out through the slot. This critical transition occurs in a matter of seconds. Once above this critical flow rate, the angle at which the interface enters the slot decreases as the effective flow rate increases.

Experimental work has shown that there is a large scatter in the values of the critical flow rate, and a theoretical investigation of the flow is important in interpreting results obtained in experiments and in the field. Solution of the Navier–Stokes equations in such problems using finite-difference or finite-element techniques is fraught with difficulties because of the moving interface, and the very rapid transitions in density and velocity within.

Craya (1949) approximated the subcritical flow (i.e. when only the lower layer is being withdrawn) as a steady irrotational motion of an inviscid incompressible fluid, and the withdrawal slot by a line sink. This assumption necessitates the solution of Laplace's equation in the fluid domain subject to conditions which ensure continuity of pressure across the interface and prohibit flow through the boundaries or across the interface. The problem is further complicated by the fact that the location of the interface is unknown. This model of the flow clearly neglects some features which may be of importance in determining the exact details of the flow in a real situation, such as viscous effects near the walls and along the interface, the time-dependent nature of the flow as the level falls, and the transient effects of opening the slot.

As a starting point for a study of these flows, the work of Craya (1949) and others (see later) over 40 years show that this model does give the correct qualitative behaviour of the flow. There is certainly room for improvement of the model by incorporating some of the effects mentioned above, perhaps using boundary-layer theory and allowing time-dependence, but the increase in difficulty of solving the resulting mathematical equations is significant.

In this paper, this ideal fluid model is extended to consider the flow when both layers are being drawn through the slot. The interface is assumed to be very thin, and once again steady irrotational flow of an incompressible inviscid fluid is considered.

The critical value of flow rate can be shown to be characterized by the Froude number in the lower layer,

$$F = (Q_1^2/g'H^3)^{1/2}, \tag{1.1}$$

where Q_1 is the flux of fluid into the slot per unit width from the lower layer, $g' = (\Delta\rho/\rho)g$ is the effective gravity, where g is gravity, $\Delta\rho$ is the density difference between the two layers, ρ is a reference density, and H is the depth of the sink beneath the equilibrium level of the interface.

Previous work on this problem can be divided into theoretical work in the subcritical regime, in which the flow is restricted to the lower (high-density) region of the fluid, and some experimental work.

It is interesting that if one assumes that the flow in the upper layer is stagnant, the equations describing the flow of a single layer of fluid beneath a free surface are identical to those describing the flow in the lower layer of a two-layer fluid beneath an interface, except that the gravity g is replaced by the effective gravity g' (see e.g. Jirka 1979; Yih 1980).

An exact solution in the subcritical regime was found by Sautreaux (1901), and subsequently by Craya (1949). This solution consists of a downward cusp in the interface directly above the sink, and occurs when there is a wall sloping downward from the sink with an angle of 30° to the horizontal. Tuck & Vanden-Broeck (1984) used a numerical series truncation method to compute a similar cusped flow for the case of a line sink in a single-fluid region confined only by a free surface. Further numerical solutions of this type were later obtained by Hocking (1985) and Vanden-Broeck & Keller (1987) for slightly different geometries, including some in which the lower layer was of finite depth. One interesting aspect of the solutions to the problem in which the fluid domain is of infinite depth is that the cusped solutions occur at a unique value of the Froude number, while in the finite-depth problem such cusped solutions occur over a continuous range of Froude numbers (Vanden-Broeck & Keller 1987; Hocking 1991 *a*).

These cusped solutions are thought to occur at the critical point in the flow, that is

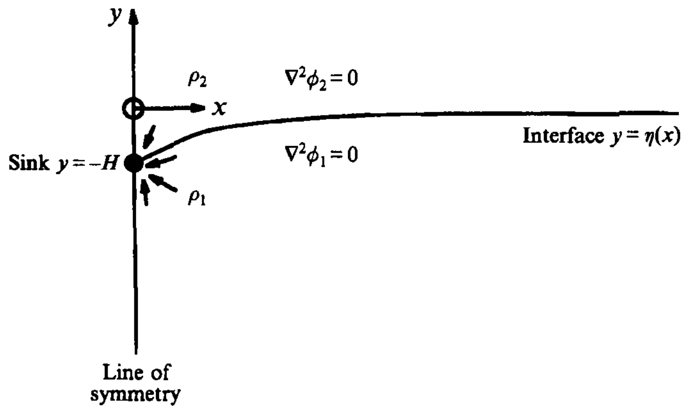

FIGURE 1. Figure defining the problem under consideration.

flows with a Froude number beneath this value are restricted to the lower layer, while those with higher values of the Froude number involve water from both layers entering the sink. In the cases where there is a continuous range of Froude numbers for which cusped solutions exist, it is most likely that the minimum value of F corresponds to the critical flow (Yih 1980). However, these cusped solutions are difficult to observe because if they exist at all the unsteady flow will pass through this state very quickly. It is even possible that viscous effects at the interface may prevent their formation. At the level of the approximations used in this work, however, it is likely that whether they exist or not, they do represent the transition point between the one- and two-layer flow regimes.

For withdrawal through a slot, i.e. the two-dimensional problem, the experimental work is restricted to that of Gariel (1949), Harleman & Elder (1965), Wood & Lai (1972) and Hocking (1991 *b*). In almost all of these cases it was found that the critical drawdown point, at which the upper fluid begins to flow out through the sink, occurs at a Froude number much lower than that predicted by the cusped solutions outlined above. This result is consistent with the experimental results for a point sink, in which the drawdown occurs at much lower values of the Froude number than expected (Harleman & Elder 1965; Jirka & Katavola 1979; Lawrence & Imberger 1979).

In this paper, an integral equation approach is used to compute accurate numerical solutions to the supercritical flow problem in which water from both layers is flowing out through the sink.

In §2 the flow into a horizontal slot is considered together with how the sink representation approximates such a flow. This is followed by non-dimensionalization of the equations, the derivation of the integral equations which must be satisfied in the two layers, and the conditions on the interface.

Section 3 describes the numerical scheme which is used to solve this system of equations, and §4 discusses the results in the context of other work on this problem. It is found that for any fixed value of the Froude number, there is a single value for the angle at which the interface enters the sink, ranging from an angle approaching zero, i.e. horizontal, as $F \rightarrow \infty$, up to an angle approaching 90°, i.e. vertical, as the value of F drops down to a critical value, F_C. The value of F_C which is obtained is very close to the unique value obtained by Tuck & Vanden-Broeck (1984) for the cusped solution in a single-layer flow. In addition, the shape of the interface approaches that of the cusped flow (see figure 4). Thus the results provide evidence in support of the hypothesis that in this model the cusped solutions represent the critical transition flows, at least for a fluid of infinite depth.

2. Problem formulation

The steady irrotational motion of an inviscid incompressible fluid in two dimensions is to be examined. The fluid is separated by an interface of infinitesimal thickness into two homogeneous regions of different density. The solutions we seek are those in which the interface is drawn down a distance H to a point where it enters the sink with an angle α to the horizontal. Fluid is being withdrawn from both above and below the interface (see figure 1).

Let $z = x+\mathrm{i}y$ be the physical plane, with the origin directly above the line sink, and at the level of the interface far away from the sink. If $y = \eta(x)$ is the equation of the interface, suppose the region below the interface to have density ρ_1 and the region above the interface to have density ρ_2. The velocity potentials of the separate flow fields below and above the interface must both satisfy Laplace's equation, i.e.

$$\left.\begin{aligned}\nabla^2\Phi_1(x,y) &= 0, \quad y < \eta(x),\\ \nabla^2\Phi_2(x,y) &= 0, \quad y > \eta(x).\end{aligned}\right\} \tag{2.1}$$

As the sink is approached, the velocity potentials must have the correct behaviour, which is

$$\left.\begin{aligned}\Phi_1 &\to -\frac{Q_1}{\frac{1}{2}\pi+\alpha}\log[x^2+(y+H)^2]^{1/2} \quad \text{as} \quad (x,y)\to(0,-H), \quad y<\eta(x)\\ \text{and}\quad \Phi_2 &\to -\frac{Q_2}{\frac{1}{2}\pi-\alpha}\log[x^2+(y+H)^2]^{1/2} \quad \text{as} \quad (x,y)\to(0,-H), \quad y>\eta(x),\end{aligned}\right\} \tag{2.2}$$

where Q_1 and Q_2 are the respective fluxes per unit width from within the two regions. There is a relationship between these two values which must hold if the dynamic condition on the interface is to be satisfied. Applying the Bernoulli equation to the streamline along the interface, and noting that for steady flow there must be no pressure difference across the interface leads to the result that

$$\rho_1 g\eta(x)+\tfrac{1}{2}\rho_1(\Phi_{1x}^2+\Phi_{1y}^2) = \rho_2 g\eta(x)+\tfrac{1}{2}\rho_2(\Phi_{2x}^2+\Phi_{2y}^2) \quad \text{on} \quad y=\eta(x). \tag{2.3}$$

This equation can be rearranged to give

$$2g'\eta(x)+((\Phi_{1x}^2+\Phi_{1y}^2)-\gamma(\Phi_{2x}^2+\Phi_{2y}^2)) = 0, \tag{2.4}$$

where $g' = [(\rho_1-\rho_2)/\rho_1]g$, and $\gamma = \rho_2/\rho_1$.

We note in passing that if the velocity in the upper layer is zero (stagnant fluid), (2.4) becomes $2g'\eta(x)+(\Phi_{1x}^2+\Phi_{1y}^2) = 0$, which is identical to the equation for constant pressure on a free surface, except that g is replaced by g'. Therefore the work done in solving free-surface flow problems relates directly to the current two-layer flow situation, and is an analogue of the subcritical flow behaviour, in which the upper fluid is assumed to be stagnant.

Considering the behaviour of the flow near the sink (2.2), we see that in order to satisfy (2.4), it is necessary that

$$\frac{Q_1}{\frac{1}{2}\pi+\alpha} = \gamma^{1/2}\frac{Q_2}{\frac{1}{2}\pi-\alpha}. \tag{2.5}$$

This result seems slightly surprising at first. However, if we consider the flow as an approximation to a flow into a thin horizontal slot which extends to $x = -\infty$, we see that a long distance into the slot, if δ_1 and δ_2 are the depths of the two layers flowing horizontally with velocities u_1 and u_2 respectively, the ratio of fluid originating from the two outer regions is, from (2.4),

$$\frac{Q_2}{Q_1} \approx \frac{u_2\,\delta_2}{u_1\,\delta_1} = \gamma^{-1/2}\frac{\delta_2}{\delta_1}\left[1-\frac{2g'(H+\delta_2)}{u_1^2}\right]^{1/2},$$

so that once again there is a condition on the flow which relates the density in the two layers with the flow at the interface and in the slot.

Further, if we assume that the size of the slot is small compared to its depth beneath the undisturbed height of the interface, i.e. $\delta_i \ll H, i = 1, 2$, then

$$\frac{Q_2}{Q_1} = \gamma^{-1/2}\frac{\delta_2}{\delta_1}\left[1-2F^{-2}\left(\frac{\delta_1}{H}\right)^2\right]^{1/2},$$

which means that

$$Q_2/Q_1 \approx \gamma^{-1/2}\,\delta_2/\delta_1.$$

Comparing this to the proposed model of flow into a line sink where the interface enters with angle α, we see that

$$\frac{Q_2}{Q_1} = \gamma^{-1/2}\frac{\pi-2\alpha}{\pi+2\alpha},$$

and thus there is a direct analogy between the angle of entry into the sink and the depth downstream in a finite-sized slot. Huber (1960) was able to show that if the two outer layers are of finite depth, there is a direct relationship between the Froude number and the angle of entry. However, in the unconfined flow considered here there is no way to determine the relationship between Froude number and angle of entry into the sink (or of depths downstream in the slot) without solving the full system of equations.

The final condition to be satisfied is that there be no flow across the interface, and we ensure that this is so by defining stream functions Ψ_1 and Ψ_2, and enforcing the condition that $\Psi_1 = \Psi_2 = 0$ along the interface, $y = \eta(x)$.

If we let $y' = y/H$, $x' = x/H$, $\Phi_1' = [2Q_1/(\pi+2\alpha)]\,\Phi_1$ and $\Phi_2' = [2Q_1\gamma^{1/2}/(\pi+2\alpha)]\,\Phi_2$, then the non-dimensional form of the dynamic condition is

$$\tfrac{1}{2}(\pi+2\alpha)^2F^{-2}\eta' + ((\Phi_{1x}')^2+(\Phi_{1y}')^2) - ((\Phi_{2x}')^2+(\Phi_{2y}')^2) = 0, \tag{2.6}$$

where

$$F = (Q_1^2/g'H^3)^{1/2}, \tag{2.7}$$

$$\left.\begin{aligned} &\text{and} \quad \Phi_1' \to -\log[(x')^2+(y'+1)^2]^{1/2} \quad \text{as} \quad (x',y') \to (0,-1), \quad y' < \eta'(x) \\ &\text{and} \quad \Phi_2' \to -\log[(x')^2+(y'+1)^2]^{1/2} \quad \text{as} \quad (x',y') \to (0,-1), \quad y' > \eta'(x). \end{aligned}\right\} \tag{2.8}$$

We will henceforth dispense with the prime notation for dimensionless variables. One method of solution to this problem is to write a complex potential for each region which builds in the correct behaviour both as the sink is approached and in the far field, and then compute the corrections to these. Choices which satisfy these requirements are

$$\left.\begin{aligned} f_1 &= \Phi_1 + \mathrm{i}\Psi_1 = -\log(z+\mathrm{i}) - \frac{2\alpha}{\pi}\log\left(z-\mathrm{i}\frac{\pi}{2\alpha}\right) + w_1, \quad y < \eta(x) \\ &\text{and} \\ f_2 &= \Phi_1 + \mathrm{i}\Psi_1 = -\log(z+\mathrm{i}) + \frac{2\alpha}{\pi}\log\left(z+\mathrm{i}\frac{\pi}{2\alpha}\right) + w_2, \quad y > \eta(x), \end{aligned}\right\} \tag{2.9}$$

where α is the angle of the interface at the point of entry into the sink and $w_j = \phi_j + \mathrm{i}\psi_j$, $j = 1, 2$, are the correction terms for the full velocity potentials. In each case these

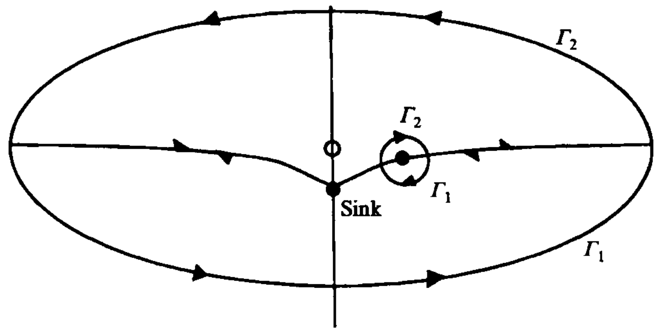

FIGURE 2. Contours used in the derivation of the integral equations (2.16).

choices represent the addition of another singular point outside the domain of interest. These are a line sink at $y = \pi/(2\alpha)$ for the lower fluid, and a line source at $y = -\pi/(2\alpha)$ for the upper fluid. It is not difficult to show that these choices satisfy the requirement that the line $\Psi_j = 0, j = 1,2$, enters the sink at an angle α to the horizontal, and that $\eta(x) \to 0$ as $x \to \infty$, provided

$$\left.\begin{aligned} \psi_1(x,\eta) &= -\arctan\left(\frac{\eta(x)+1}{x}\right) - \frac{2\alpha}{\pi}\arctan\left(\frac{\eta(x)-\pi/2\alpha}{x}\right) \\ \text{and}\qquad \psi_2(x,\eta) &= -\arctan\left(\frac{\eta(x)+1}{x}\right) + \frac{2\alpha}{\pi}\arctan\left(\frac{\eta(x)+\pi/2\alpha}{x}\right). \end{aligned}\right\} \tag{2.10}$$

This choice of f_1 and f_2 also ensures that $w_j \to 0, j = 1,2$, as $|z| \to \infty$ and as $z \to -\mathrm{i}$. The functions

$$\left.\begin{aligned} w_1(x,y) &= \phi_1(x,y) + \mathrm{i}\psi_1(x,y), \quad y < \eta(x), \\ w_2(x,y) &= \phi_2(x,y) + \mathrm{i}\psi_2(x,y), \quad y > \eta(x) \end{aligned}\right\} \tag{2.11}$$

must be analytic in their respective domains. Following Forbes (1985), we apply Cauchy's Theorem to $w_j, j = 1,2$, on the regions above and below the interface, to get

$$\pi w_1(z_0) = \int_{\Gamma_1} \frac{w_1(z)}{z-z_0}\mathrm{d}z, \qquad \pi w_2(z_0) = \int_{\Gamma_2} \frac{w_2(z)}{z-z_0}\mathrm{d}z, \tag{2.12}$$

where Γ_1 and Γ_2 are the contours shown in figure 2, and z_0 lies on the boundary in each case. Now since $w_1(z)$ and $w_2(z) \to 0$ as $|z| \to \infty$, the contribution of that part of Γ which consists of the circular arc can be shown to be zero. Thus we only need to integrate along the interface. If we let s be the arclength along the interface starting from the sink, so that

$$\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}\eta}{\mathrm{d}s}\right)^2 = 1, \tag{2.13}$$

we obtain

$$\pi \mathrm{i} w_1(z(s)) = \int_{-\infty}^{\infty} \frac{w_1(z(t))\,\mathrm{d}z/\mathrm{d}t}{z(t)-z(s)}\mathrm{d}t, \qquad -\pi\mathrm{i} w_2(z(s)) = \int_{-\infty}^{\infty} \frac{w_2(z(t))\,\mathrm{d}z/\mathrm{d}t}{z(t)-z(s)}\mathrm{d}t. \tag{2.14}$$

Since ψ_1, ψ_2 are known along the interface from (2.10), these represent integral equations for ϕ_1 and ϕ_2 respectively. Taking the real parts, and utilizing the symmetry of the situation about the line $x = 0$, i.e.

$$\left.\begin{aligned} x(-s) = -x(s), \quad y(-s) = y(s), \quad x'(-s) = x'(s), \quad y'(-s) = -y'(s), \\ \phi_j(-s) = \phi_j(s), \quad \psi_j(-s) = -\psi_j(s), \quad j = 1,2 \end{aligned}\right\} \tag{2.15}$$

the integral equations become

$$\phi_j(s) = \frac{\kappa_j}{\pi}\int_0^\infty \phi_j(t)\left(\left[\frac{x'(t)\,\Delta y - y'(t)\,\Delta x}{\Delta x^2+\Delta y^2}\right] - \left[\frac{x'(t)\,\Delta y - y'(t)\,\Delta x_+}{\Delta x_+^2+\Delta y^2}\right]\right)$$
$$+\psi_j(t)\left(\left[\frac{x'(t)\,\Delta x - y'(t)\,\Delta y}{\Delta x^2+\Delta y^2}\right] - \left[\frac{x'(t)\,\Delta x_+ - y'(t)\,\Delta y}{\Delta x_+^2+\Delta y^2}\right]\right)\mathrm{d}t, \quad j = 1,2, \quad (2.16)$$

where $\Delta x = x(t)-x(s), \Delta x_+ = x(t)+x(s)$ and $\Delta y = y(t)-y(s)$, and $\kappa_1 = +1$, $\kappa_2 = -1$. Using the arclength formulation, the interface condition (2.6) simplifies a little to become

$$\tfrac{1}{2}(\pi+2\alpha)^2 F^{-2}\eta(s) + \Phi_{1s}^2 - \Phi_{2s}^2 = 0. \quad (2.17)$$

Thus the problem which must be solved is the combination of the two integral equations given by (2.16) and the interface condition (2.17) along the unknown interface, $y = \eta(x)$.

3. Solution method

As the location of the interface is unknown, and because of the quadratic dependence of the interface condition upon the velocity, this is a highly nonlinear problem. In order to solve this problem we must resort to a numerical scheme. There is an additional complication caused by the presence of the singular point representing the sink flow on the interface itself, which makes the problem numerically unstable near to the sink if it is not treated very carefully. An algorithm which was found to be successful, however, is described below.

(i) Make a guess for $\eta'(s)$, the rate of change of η with respect to the arclength, at a set of evenly spaced points along the interface, s_k, $k = 1,2,\ldots,N$, and also make a guess for the entry angle of the interface into the sink α, given a fixed value of F.

(ii) This guess for $\eta'(s)$ can be integrated to give $\eta(s)$, and noting that

$$x'(s) = [1-\eta'(s)^2]^{1/2}, \quad (3.1)$$

$x(s)$ can also be obtained by integration. A trapezoidal rule integration scheme was found to be adequate for these calculations.

(iii) Using x, η, $x'(s)$, $\eta'(s)$ along the interface, the integral equations (2.16) for ϕ_1 and ϕ_2 can be solved by making a guess for ϕ_1 and ϕ_2 at the same set of points, i.e. s_k, $k = 1,2,\ldots,N$, and using a Newton iteration scheme. The accuracy of the numerical integration is crucial to the solution of the full problem. The singular part of the principal value integral was removed by noting that

$$\int_0^{z_N}\frac{w_j(z)}{z-z_0}\,\mathrm{d}z = \int_0^{z_N}\frac{w_j(z)-w_j(z_0)}{z-z_0}\,\mathrm{d}z + w_j(z_0)\ln\left[\frac{z_N-z_0}{z_0}\right], \quad (3.2)$$

where z_N corresponds to the point at which the integral is truncated. It was found that provided the truncation point was chosen to be greater than $s_N \approx 7$ the differences in the results were minimal. Also, it is essential to include an approximation to the portion of the integral which is neglected. This correction term was found to be very important in the success of the method. Both ϕ and ψ can be shown to behave like $O(s^{-1})$ as $s\to\infty$, so if we let $\phi = \phi_N s_N/s$, and $\psi = \psi_N s_N/s$ for $s > s_N$, and note that for large values of s, $x \approx x_N+(s-s_N)$ and $y \approx 0$, then a simple correction term can be added to each integral term in the integral equations. A Gill–Miller finite-difference scheme from the NAG mathematical function library was used to perform the integrations.

(iv) Once ϕ_1 and ϕ_2 have been obtained, a forward difference scheme can be used to

F	α
7.823	0.385
6.117	0.468
4.379	0.619
2.575	1.004
2.336	1.079
2.013	1.305
1.958	1.352
1.905	1.400

TABLE 1. Computed Froude number for different angles of entry of the interface into the sink

calculate ϕ_{1s} and ϕ_{2s}, and the error in the interface condition (2.17) can be evaluated. If the error is small at all points on the interface, say less than 10^{-9}, then the algorithm is stopped. If the error is greater than this value at any point, then a Newton's method is used to update the original guess for $\eta'(s)$, and we return to step (ii).

Although the convergence of the scheme was rapid, taking about six iterations at each new value of the Froude number, the need for the integral equations to be solved numerically N^2 times within each iteration meant that a long time was taken for each simulation. Values of N up to 100 were used, but the time taken to use larger values of N was found to be prohibitive.

It is to be expected that for a given value of Froude number, F, there would be a single value of entry angle α for which a solution would exist, since if the withdrawal rate is higher, the interface would be pulled down more. This was found to be the case. Attempts to compute solutions in which both F and α were specified were a failure, as were attempts to compute solutions in which both were free parameters.

A solution was first obtained at a large value of F for which the interface was almost horizontal, and then successive solutions were obtained by decreasing the value of F and using the previous solution as a starting guess for the iteration scheme. As F decreased, α was found to increase, as shown in table 1, which shows only some of the results obtained, and figure 3.

It was not possible to obtain solutions for values of α greater than 1.4, since for larger values of α, the method failed to converge. Remembering that $\alpha = 0.5\pi \approx 1.57$ coincides with the case in which the flow is restricted to the lower layer, it seems likely that the failure was due to the high curvature of the interface as the critical flow is approached. As N was increased it was found that higher values of α could be obtained before the method failed, but the time taken became prohibitive for the reason mentioned above. However, solutions with $N = 80$ were found to be accurate to three decimal places except near the point at which the numerical scheme failed, and this value of N was used for most calculations.

4. Results and comment

The major result of this paper is shown in figure 3. The solid line represents a least-squares fit of the relationship of F to α obtained using the numerical solutions, shown as crosses. As can be seen, this curve, given by

$$F = 0.5105(\pi + 2\alpha)\,\alpha^{-1.419}, \tag{4.1}$$

fits the data very well. The $(\pi + 2\alpha)$ term in this equation is due to the change in flux as the angle of entry into the sink changes. In order to compare these results with the

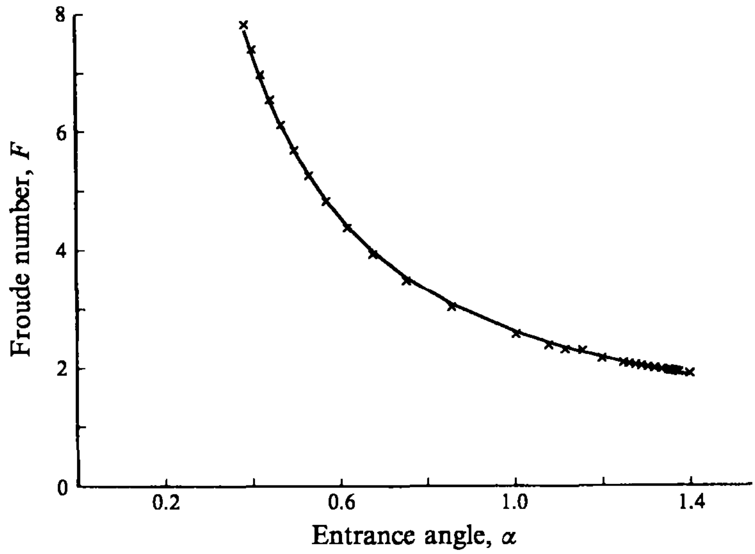

FIGURE 3. Plot of the Froude number F against the angle of entry of the interface into the sink, α. The solid line is a best-fit curve and the crosses are solutions obtained using the numerical scheme.

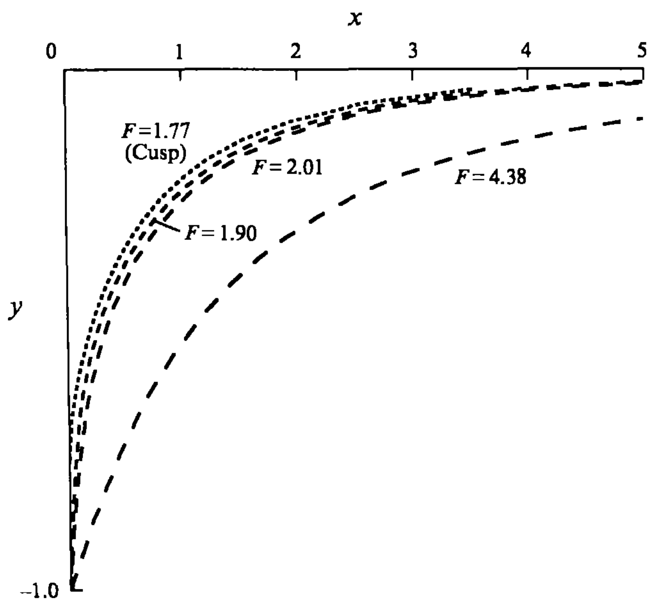

FIGURE 4. Interface shapes for different values of α compared with the cusp solution of Tuck & Vanden-Broeck (1984).

cusp solution of Tuck & Vanden-Broeck (1984), a similar curve was computed for all values of Froude number $F < 2.2$ and the limit was taken as $\alpha \to \frac{1}{2}\pi$. In this manner, a critical value of $F_C \approx 1.73$ was obtained. This compares extremely well with the value $F_C = 1.77$ calculated by Tuck & Vanden-Broeck (1984) for the unique cusped solution. This is very strong evidence that this cusped solution is indeed the critical value at which the transition between single-layer and two-layer flow occurs.

Further evidence is provided in figure 4, which shows the cusped solution of Tuck

& Vanden-Broeck (1984) compared with the interface shapes computed for increasing values of α in the work described in this paper.

This paper describes a method which was used to compute accurate numerical solutions to the problem of supercritical withdrawal through a line sink from a two-layer fluid. The results support the belief that the cusped solutions of Craya (1949), Tuck & Vanden-Broeck (1984), and Hocking (1985), are the transition between single- and two-layer flows. It therefore seems likely that the differences between the numerical solutions and the experimentally determined critical flow values are due to the effect of the non-zero thickness of the interface, viscous effects, and transient effects of turning on the sink.

In a vertically confined fluid however, it has been shown on numerous occasions (Vanden-Broeck & Keller 1987; Hocking 1988, 1991 *a*) that there are cusped solutions over a continuous range of Froude numbers for a given geometry. For some cases there is also a unique subcritical solution with a cusp, and it is possible that this solution represents the transition from the one-layer to the two-layer flows. However, not all ratios of sink to base depth have these unique solutions. This raises the question as to which of this range of solutions is the true critical value. Work is under way to resolve this issue.

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